

REDUCED LIMIT FOR SEMILINEAR BOUNDARY VALUE PROBLEMS WITH MEASURE DATA.

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1. Introduction

In this article we consider equations of the type

$$(1.1) \quad \begin{aligned} -\Delta u + g \circ u &= \mu & \text{in } \Omega, \\ u &= \nu & \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded C^2 domain in \mathbb{R}^N , g is defined on $\Omega \times \mathbb{R}$ and $g \circ u(x) = g(x, u(x))$.

We assume that the nonlinearity g satisfies the conditions,

$$(1.2) \quad \begin{aligned} (a) \quad &g(x, \cdot) \in C(\mathbb{R}), \quad g(x, 0) = 0, \\ (b) \quad &g(x, \cdot) \text{ is a non decreasing and odd function,} \\ (c) \quad &g(\cdot, t) \in L^1(\Omega, \rho) \quad \forall t \in \mathbb{R} \end{aligned}$$

where

$$\rho(x) = \text{dist}(x, \partial\Omega).$$

The family of functions satisfying these conditions will be denoted by $\mathcal{G}_0 = \mathcal{G}_0(\Omega)$.

With respect to the data, we assume that $\nu \in \mathcal{M}(\partial\Omega)$ and $\mu \in \mathcal{M}(\Omega, \rho)$ where:

$\mathcal{M}(\partial\Omega)$ denotes the space of bounded Borel measure on $\partial\Omega$ with the usual total variation norm.

$\mathcal{M}(\Omega, \rho)$ denotes the space of signed Radon measure μ in Ω such that

$$\rho\mu \in \mathcal{M}(\Omega) \quad \text{where} \quad \rho(x) := \text{dist}(x, \partial\Omega).$$

The norm of a measure $\mu \in \mathcal{M}(\Omega, \rho)$ is given by

$$\|\mu\|_{\Omega, \rho} = \int_{\Omega} \rho \, d|\mu|.$$

$L^1(\Omega, \rho)$ denotes the weighted Lebesgue space with weight ρ .

We say that $u \in L^1(\Omega)$ is a weak solution of (1.1) if $g \circ u \in L^1(\Omega, \rho)$ and u satisfies the the following,

$$(1.3) \quad \int_{\Omega} (-u\Delta\phi + (g \circ u)\phi) \, dx = \int_{\Omega} \phi \, d\mu - \int_{\partial\Omega} \frac{\partial\phi}{\partial\mathbf{n}} \, d\nu \quad \forall \phi \in C_0^2(\bar{\Omega}).$$

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where

$$C_0^2(\bar{\Omega}) := \{\phi \in C^2(\bar{\Omega}) : \phi = 0 \text{ on } \partial\Omega\}.$$

Definition 1.1. Given $g \in \mathcal{G}_0$, we denote by $\mathcal{M}^g(\Omega)$ the set of all measures $\mu \in \mathcal{M}(\Omega, \rho)$ such that the boundary value problem

$$(1.4) \quad -\Delta u + g \circ u = \mu \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega$$

possesses a weak solution. If $\mu \in \mathcal{M}^g(\Omega)$, we say that μ is a good measure in Ω .

We denote by $\mathcal{M}^g(\partial\Omega)$ the set of all measures $\nu \in \mathcal{M}(\partial\Omega)$ such that the boundary value problem

$$(1.5) \quad -\Delta u + g \circ u = 0 \text{ in } \Omega; \quad u = \nu \text{ on } \partial\Omega$$

possesses a weak solution. If $\nu \in \mathcal{M}^g(\partial\Omega)$, we say that ν is a good measure on $\partial\Omega$.

Finally, the set of pairs of measures $(\mu, \nu) \in \mathcal{M}(\Omega, \rho) \times \mathcal{M}(\partial\Omega)$ such that (1.1) possesses a solution will be denoted by $\mathcal{M}^g(\bar{\Omega})$.

In a recent paper Marcus and Ponce [3] studied the following problem. Let $\{\mu_n\} \subset \mathcal{M}^g(\Omega)$ be a weakly convergent sequence: $\mu_n \rightharpoonup \mu$ relative to $C_0(\bar{\Omega})$. Let u_n be the solution of (1.4) with $\mu = \mu_n$ and assume that $u_n \rightarrow u$ in $L^1(\Omega)$. In general u does not satisfy (1.4). But it was shown that there exists a measure $\mu^\#$ such that

$$(1.6) \quad -\Delta u + g \circ u = \mu^\#.$$

Moreover this measure depends only on the fact that u_n satisfies the equation

$$-\Delta u_n + g \circ u_n = \mu_n.$$

It is independent of the boundary data for u_n or indeed on whether u_n has a measure boundary trace. If $\mu_n \geq 0$ then

$$0 \leq \mu^\# \leq \mu.$$

Furthermore it was shown that, under a mild additional condition on g , the following result holds:

Let v_n be the solution of the Dirichlet problem for equation $-\Delta v = \mu_n$. Suppose that $\mu_n \geq 0$ and that $\{g \circ v_n\}$ is bounded in $L^1(\Omega; \rho)$. Then $\mu^\#$ and μ are mutually a.c.

The measure $\mu^\#$ was called the *reduced limit* of $\{\mu_n\}$. This notion is in some sense related to the notion of ‘reduced measure’ introduced in [2]. For a specific choice of $\{\mu_n\}$ the reduced limit $\mu^\#$ coincides with the reduced measure. However in general they are not equal.

In the present paper we continue this study considering similar questions with respect to sequences of pairs $\{(\mu_n, \nu_n)\} \subset \mathcal{M}^g(\bar{\Omega})$. Such a sequence is called a g good sequence. Suppose that $\nu_n \rightharpoonup \nu$ (weak convergence in $\mathcal{M}(\partial\Omega)$) and that $\rho\mu_n \rightharpoonup \tau$ (weak convergence in $\mathcal{M}(\bar{\Omega})$). We shall say that (τ, ν) is the weak limit of the sequence.

Let u_n be the solution of (1.1) with $(\mu, \nu) = (\mu_n, \nu_n)$ and suppose that $u_n \rightarrow u$ in $L^1(\Omega)$. By [3], u satisfies equation (1.6). Here we show that there exists a measure $\nu^* \in \mathcal{M}(\partial\Omega)$ such that u is the weak solution of the boundary value problem

$$\begin{aligned} -\Delta u + g \circ u &= \mu^\# \quad \text{in } \Omega \\ u &= \nu^* \quad \text{on } \partial\Omega. \end{aligned}$$

The pair $(\mu^\#, \nu^*)$ is called the *reduced limit* of $\{(\mu_n, \nu_n)\}$.

In general ν^* depends on the sequence of pairs $\{(\mu_n, \nu_n)\}$, not only on $\{\nu_n\}$. If g is subcritical we show that

$$\nu^* = \nu + \tau \mathbf{1}_{\partial\Omega}.$$

However in general the dependence of ν^* on the sequence of pairs is much more complex.

Here are some of our main results.

Theorem 1.2. *Suppose that $\mu_n \geq 0$ and $\nu_n \geq 0$ for every $n \geq 1$. If $\nu^\#$ is the reduced limit of the sequence $\{(\mu_n, \nu_n)\}$ then*

$$0 \leq \nu^\# \leq \nu \quad \text{and} \quad 0 \leq \nu^* \leq \nu^\# + \tau \mathbf{1}_{\partial\Omega}.$$

Theorem 1.3. *Let v_n denote the solution of*

$$-\Delta v = \mu_n \quad \text{in } \Omega, \quad v = \nu_n \quad \text{on } \partial\Omega.$$

Suppose that:

- (i) $\mu_n \geq 0$ and $\nu_n \geq 0$ for every $n \geq 1$,
- (ii) $\{g \circ v_n\}$ is bounded in $L^1(\Omega; \rho)$,
- (iii) $g \in \mathcal{G}_0$ satisfies the condition

$$\lim_{a, t \rightarrow \infty} \frac{g(x, at)}{ag(x, t)} = \infty \quad \text{uniformly with respect to } x \in \Omega.$$

Then $\nu + \tau \mathbf{1}_{\partial\Omega}$ and ν^ are mutually a.c. In particular ν and $\nu^\#$ are mutually a.c.*

For the statement of the next result we need an additional definition.

Definition 1.4. *A nonnegative measure $\sigma \in \mathcal{M}(\partial\Omega)$ is g -negligible if*

$$\{\lambda \in \mathcal{M}(\partial\Omega) : 0 < \lambda \leq \sigma\} \cap \mathcal{M}^g(\partial\Omega) = \emptyset.$$

Theorem 1.5. *Assume that $g \in \mathcal{G}_0$ is convex and satisfies the Δ_2 condition. Let $\{(\mu_n, \nu_n)\}$ and $\{(\tilde{\mu}_n, \nu_n)\}$ be g -good sequences with weak limits (τ, ν) and $(\tilde{\tau}, \nu)$ respectively. Assume that, for every $n \geq 1$, $(|\mu_n|, |\nu_n|)$ and $(|\tilde{\mu}_n|, |\nu_n|)$ are in $\mathcal{M}^g(\bar{\Omega})$.*

Let u_n (resp \tilde{u}_n) be the solution of (1.1) with $(\mu, \nu) = (\mu_n, \nu_n)$ (resp. $(\tilde{\mu}_n, \nu_n)$). Assume that

$$u_n \rightarrow u, \quad \tilde{u}_n \rightarrow \tilde{u} \quad \text{in } L^1(\Omega)$$

and let (μ^, ν^*) and $(\tilde{\mu}^*, \tilde{\nu}^*)$ denote the reduced limits of $\{(\mu_n, \nu_n)\}$ and $\{(\tilde{\mu}_n, \nu_n)\}$ respectively.*

If a subsequence of $\{\rho|\tilde{\mu}_n - \mu_n|\}$ converges weakly in $\mathcal{M}(\bar{\Omega})$ to a measure Λ such that $\Lambda|_{\partial\Omega}$ is negligible then

$$\nu^* = \tilde{\nu}^*.$$

2. DEFINITIONS AND AUXILLIARY RESULTS.

Definition 2.1. Let $g \in \mathcal{G}_0$. We say that g satisfies the Δ_2 condition if there exists a constant $c > 0$ such that

$$g(x, a+b) \leq c(g(x, a) + g(x, b)) \quad \forall x \in \Omega, \quad a > 0, \quad b > 0.$$

In the next proposition we gather some classical results concerning the boundary value problem (1.1).

Proposition 2.2. Suppose that $g \in \mathcal{G}_0$, $\nu \in \mathcal{M}(\partial\Omega)$ and $\mu \in \mathcal{M}(\Omega; \rho)$. Then:

(i) If $\mu, \nu \geq 0$ then $(\mu, \nu) \in \mathcal{M}^g(\bar{\Omega}) \implies \mu \in \mathcal{M}^g(\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$.

(ii) If $\mu, \nu \geq 0$, g satisfies Δ_2 condition then

$$\mu \in \mathcal{M}^g(\Omega) \quad \text{and} \quad \nu \in \mathcal{M}(\partial\Omega) \implies (\mu, \nu) \in \mathcal{M}^g(\bar{\Omega}).$$

(iii) Assume that $(\mu, \nu) \in \mathcal{M}^g(\bar{\Omega})$. Then (1.1) possesses a unique solution u . This solution satisfies:

$$(2.1) \quad \|u\|_{L^1(\Omega)} + \|g \circ u\|_{L^1(\Omega, \rho)} \leq C(\|\mu\|_{\mathcal{M}(\Omega; \rho)} + \|\nu\|_{\mathcal{M}(\partial\Omega)}),$$

where C is a constant depending only on Ω .

(iv) Under the assumption of part (ii), $u \in L^p(\Omega)$ for $1 \leq p < \frac{N}{N-1}$ and there exists a constant $C(p)$ depending only on p and Ω such that

$$(2.2) \quad \|u\|_{L^p(\Omega)} \leq C(p)(\|\mu\|_{\mathcal{M}(\Omega; \rho)} + \|\nu\|_{\mathcal{M}(\partial\Omega)}).$$

Furthermore, $u \in W_{loc}^{1,p}(\Omega)$ and for every domain $\Omega' \Subset \Omega$, there exists a constant $C(p, \Omega')$ depending on p , Ω' and Ω such that

$$(2.3) \quad \|u\|_{W^{1,p}(\Omega')} \leq C(p, \Omega')(\|\mu\|_{\mathcal{M}(\Omega')} + \|\nu\|_{\mathcal{M}(\partial\Omega)}).$$

Assertion (i) and (ii) are obvious (see e.g. [4, Theorem 2.4.5]).

Assertion (iii) is due to Brezis [1]; a proof can be found in [4] or [5].

Assertion (iv) is a consequence of (ii) and classical estimates for the corresponding linear problem.

Definition 2.3. A sequence $\{\mu_n\} \in \mathcal{M}(\Omega, \rho)$ is tight if for every $\epsilon > 0$, there exists a neighborhood U of $\partial\Omega$ such that

$$\int_{U \cap \Omega} \rho d|\mu_n| \leq \epsilon.$$

Definition 2.4. A sequence $\{\Omega_n\}$ is an exhaustion of Ω if $\bar{\Omega}_n \subset \Omega_{n+1}$ and $\Omega_n \uparrow \Omega$. We say that an exhaustion $\{\Omega_n\}$ is of class C^2 if each domain Ω_n is of this class. If, in addition, Ω is a C^2 domain and the sequence of domains $\{\Omega_n\}$ is uniformly of class C^2 , we say that $\{\Omega_n\}$ is a uniform C^2 exhaustion.

Definition 2.5. Let $u \in W_{loc}^{1,p}(\Omega)$ for some $p > 1$. We say that u possesses an M -boundary trace on $\partial\Omega$ if there exists $\nu \in \mathcal{M}(\partial\Omega)$ such that, for every uniform C^2 exhaustion $\{\Omega_n\}$ and every $h \in C(\bar{\Omega})$,

$$\int_{\partial\Omega_n} u|_{\partial\Omega_n} h \, dS \rightarrow \int_{\partial\Omega} h \, d\nu$$

where $u|_{\partial\Omega_n}$ denotes the Sobolev trace, $dS = d\mathbb{H}^{N-1}$ and \mathbb{H}^{N-1} denotes $(N-1)$ dimensional Hausdorff measure. The M -boundary trace ν is denoted by $\text{tr } \nu$.

Proposition 2.6. Let $\mu \in \mathcal{M}(\Omega, \rho)$ and $\nu \in \mathcal{M}(\partial\Omega)$. Then a function $u \in L^1(\Omega)$, with $g \circ u \in L^1(\Omega, \rho)$, satisfies (1.3) if and only if

$$\begin{aligned} -\Delta u + g \circ u &= \mu \quad (\text{in the sense of distribution}) , \\ \text{tr } u &= \nu \quad (\text{in the sense of Definition 2.5}) . \end{aligned}$$

This is an immediate consequence of [4, Proposition 1.3.7].

3. Reduced limit of a sequence of measures in $\mathcal{M}^g(\partial\Omega)$

In this section we discuss a sequence of problems

$$(3.1) \quad \begin{aligned} -\Delta u_n + g \circ u_n &= 0 \quad \text{in } \Omega, \\ u_n &= \nu_n \quad \text{on } \partial\Omega, \end{aligned}$$

where $g \in \mathcal{G}_0$ and

$$(3.2) \quad \begin{aligned} (i) \quad &\nu_n \text{ is a good measures on } \partial\Omega \quad \forall n \in \mathbb{N} \\ (ii) \quad &\nu_n \rightharpoonup \nu \text{ in } \mathcal{M}(\partial\Omega) \end{aligned}$$

Lemma 3.1. Assume that $\{\nu_n\}$ satisfies (3.2) and let u_n be the solution of (3.1). Then there exists a subsequence $\{u_{n_k}\}$ that converges in $L^1(\Omega)$.

Proof. By Proposition 2.2, $\{u_n\}$ is bounded in $L^p(\Omega)$ for every $p \in [1, N/(N-1))$. Consequently, for each such p , $\{u_n\}$ is uniformly integrable in $L^p(\Omega)$. Furthermore $\{u_n\}$ is bounded in $W_{loc}^{1,p}(\Omega)$ for every $p \in [1, N/(N-1))$. Therefore there is a subsequence $\{u_{n_k}\}$ which converges in $L_{loc}^1(\Omega)$ and point-wise a.e. to a function u . Combining these facts we conclude that $u_{n_k} \rightarrow u$ in $L^1(\Omega)$. \square

To simplify the presentation we introduce the following:

Definition 3.2. (i) Let $\{\mu_n\}$ be a bounded sequence of measures in $\mathcal{M}(\Omega; \rho)$. Assume that $\rho\mu_n$ is extended to a Borel measure $(\mu_n)_\rho \in \mathcal{M}(\bar{\Omega})$ defined as zero on $\partial\Omega$. We say that $\{\rho\mu_n\}$ converges weakly in $\bar{\Omega}$ to a measure $\tau \in \mathcal{M}(\bar{\Omega})$ if $\{(\mu_n)_\rho\}$ converges weakly to τ in $\mathcal{M}(\bar{\Omega})$, i.e.

$$\int_{\Omega} \phi \rho \, d\mu_n \rightarrow \int_{\bar{\Omega}} \phi \, d\tau \quad \forall \phi \in C(\bar{\Omega}).$$

This convergence is denoted by

$$\rho\mu_n \xrightarrow{\bar{\Omega}} \tau.$$

(ii) Let $\{\mu_n\}$ be a sequence in $\mathcal{M}_{loc}(\Omega)$. We say that the sequence converges weakly to $\mu \in \mathcal{M}_{loc}(\Omega)$ if it converges in the distribution sense, i.e.,

$$\int_{\Omega} \phi d\mu_n \rightarrow \int_{\Omega} \phi d\mu \quad \forall \phi \in C_c(\Omega).$$

This convergence is denoted by $\mu_n \xrightarrow[d]{} \mu$.

If $\{\rho\mu_n\}$ converges weakly in $\bar{\Omega}$ to τ then

$$\mu_n \xrightarrow[d]{} \mu_{int} := \frac{\tau}{\rho} \mathbf{1}_{\Omega}.$$

Thus, for τ as in part (i),

$$(3.3) \quad \tau = \tau \mathbf{1}_{\partial\Omega} + \rho\mu_{int}.$$

Lemma 3.3. Let $\{\mu_n\}$ be as in Definition 3.2(i) and assume that $\rho\mu_n \xrightarrow{\bar{\Omega}} \tau$.

Then

$$(3.4) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \varphi d\mu_n = \int_{\Omega} \varphi d\mu_{int} - \int_{\partial\Omega} \frac{\partial\varphi}{\partial\mathbf{n}} d\tau$$

for every $\varphi \in C_0^1(\bar{\Omega})$.

Proof. Put

$$(3.5) \quad \bar{\varphi} = \begin{cases} \varphi/\rho & \text{in } \Omega \\ -\frac{\partial\varphi}{\partial\mathbf{n}} & \text{on } \partial\Omega. \end{cases}$$

Then $\bar{\varphi} \in C(\bar{\Omega})$ and consequently, using (3.3),

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} \varphi d\mu_n &= \lim_{n \rightarrow \infty} \int_{\Omega} \bar{\varphi} \rho d\mu_n = \int_{\bar{\Omega}} \bar{\varphi} d\tau \\ &= \int_{\Omega} \varphi d\mu_{int} - \int_{\partial\Omega} \frac{\partial\varphi}{\partial\mathbf{n}} d\tau. \end{aligned}$$

□

Theorem 3.4. Assume that $g \in \mathcal{G}_0$ and that $\{\nu_n\}$ is a sequence of measures satisfying (3.2). Let u_n be the solution of (3.1) and assume that

$$(3.6) \quad u_n \rightarrow u \quad \text{in } L^1(\Omega).$$

Then there exists a measure $\nu^{\#} \in \mathcal{M}^g(\partial\Omega)$ such that

$$(3.7) \quad \begin{aligned} -\Delta u + g \circ u &= 0 & \text{in } \Omega, \\ u &= \nu^{\#} & \text{on } \partial\Omega. \end{aligned}$$

Furthermore $\{(g \circ u_n)\rho\}$ converges weakly in $\bar{\Omega}$ to a measure $\lambda \in \mathcal{M}(\bar{\Omega})$ and

$$(3.8) \quad \nu^{\#} = \nu - \lambda \mathbf{1}_{\partial\Omega}.$$

If $\nu_n \geq 0$ then $0 \leq \nu^\# \leq \nu$.

Remark. The measure $\nu^\#$ defined above is called *the reduced limit of $\{\nu_n\}$* . We emphasize that $\nu^\#$ depends on the sequence, not only on its limit.

Proof. By assumption $-\Delta u_n + g \circ u_n = 0$ in Ω and $u_n \rightarrow u$ in $L^1(\Omega)$. Therefore, by [3, Thm. 1.3],

$$(3.9) \quad -\Delta u + g \circ u = 0 \quad \text{in } \Omega.$$

(Note that, in the notation of [3], the present case corresponds to $\mu_n = 0$ and therefore $\mu^\# = 0$.)

Consider a subsequence of $\{u_n\}$ such that $\{\rho g \circ u_n\}$ converges weakly in $\bar{\Omega}$. The subsequence is still denoted by $\{u_n\}$ and we denote by λ the weak limit of $\{\rho g \circ u_n\}$ in $\mathcal{M}(\bar{\Omega})$. Put

$$(3.10) \quad \lambda_{in} = \lambda \mathbf{1}_\Omega, \quad \lambda_{bd} = \lambda \mathbf{1}_{\partial\Omega}.$$

Then $g \circ u_n \xrightarrow{d} \lambda_{in}/\rho$ and consequently

$$-\Delta u + \frac{\lambda_{in}}{\rho} = 0 \quad \text{in } \Omega.$$

Comparing with (3.9) we obtain,

$$(3.11) \quad \lambda_{in} = \rho(g \circ u).$$

For every $\varphi \in C_0^2(\bar{\Omega})$,

$$(3.12) \quad -\int_{\Omega} u_n \Delta \varphi \, dx + \int_{\Omega} (g \circ u_n) \varphi \, dx = -\int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} \, d\nu_n.$$

By the definition of λ , Lemma 3.3 and (3.11),

$$(3.13) \quad \lim_{n \rightarrow \infty} \int_{\Omega} (g \circ u_n) \varphi \, dx = \int_{\Omega} (g \circ u) \varphi \, dx - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} \, d\lambda$$

Therefore, taking the limit in (3.12) we obtain

$$-\int_{\Omega} u \Delta \varphi \, dx + \int_{\Omega} (g \circ u) \varphi \, dx - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} \, d\lambda = -\int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} \, d\nu.$$

Thus u is a weak solution of (3.7) with $\nu^\#$ given by (3.8). By Proposition 2.6, $\nu^\#$ is the M-boundary trace of u ; hence $\nu^\#$ is independent of the specific subsequence of $\{(g \circ u_n)\rho\}$ that converges weakly in $\bar{\Omega}$. This fact and (3.8) imply that λ_{bd} is independent of the subsequence. By (3.11), λ_{in} is independent of the subsequence. Therefore the full sequence $\{\rho(g \circ u_n)\}$ converges to λ .

If $\nu_n \geq 0$ then $u_n \geq 0$ and $g \circ u_n \geq 0$. Therefore, in this case, $\lambda \geq 0$ and consequently $\nu^\# \leq \nu$. Further, $u \geq 0$ and therefore its M-boundary trace, namely $\nu^\#$, is non-negative. \square

Lemma 3.5. *Let $\{\nu_n\}$ and $\{\nu'_n\}$ be sequences of measures in $\mathcal{M}^g(\partial\Omega)$ with weak limits ν and ν' respectively. Let u_n (resp. u'_n) be the solution of (1.1) with $\mu = 0$ and $\nu = \nu_n$ (resp. $\nu = \nu'_n$). Assume that $u_n \rightarrow u$ and $u'_n \rightarrow u'$ in $L^1(\Omega)$.*

If $\nu_n \leq \nu'_n$ for every n then $\nu^\#$ and $(\nu')^\#$ (the reduced limits of the two sequences) satisfy

$$(3.14) \quad 0 \leq (\nu')^\# - \nu^\# \leq \nu' - \nu.$$

Proof. Since $\nu_n \leq \nu'_n$ we have $u_n \leq u'_n$. Hence

$$\nu^\# = \text{tr } u \leq \text{tr } u' = (\nu')^\#$$

and

$$\lambda = \lim \rho(g \circ u_n) \leq \lim \rho(g \circ u'_n) = \lambda'.$$

By Theorem 3.4 these limits exist in the sense of weak convergence in $\mathcal{M}(\bar{\Omega})$. Furthermore,

$$\nu^\# = \nu - \lambda_{bd}, \quad (\nu')^\# = \nu' - \lambda'_{bd}.$$

Hence

$$(\nu')^\# - \nu^\# = (\nu' - \nu) - (\lambda'_{bd} - \lambda_{bd}) \leq \nu' - \nu.$$

□

Theorem 3.6. *In addition to the assumptions of Theorem 3.4, assume that g satisfies*

$$(3.15) \quad \lim_{a,t \rightarrow \infty} \frac{g(x, at)}{ag(x, t)} = \infty \quad \text{uniformly with respect to } x \in \Omega.$$

Put $v_n := \mathbb{P}(\nu_n)$, i.e.

$$(3.16) \quad -\Delta v_n = 0 \quad \text{in } \Omega, \quad v_n = \nu_n \quad \text{on } \partial\Omega.$$

If $\nu_n \geq 0$ and $\{g \circ v_n\}$ is bounded in $L^1(\Omega; \rho)$ then ν and $\nu^\#$ (the reduced limit of $\{\nu_n\}$) are mutually absolutely continuous.

We postpone the proof to Section 3 where we present a more general version of this result.

Proposition 3.7. *Assume that $g \in \mathcal{G}_0$. Let $\{\nu_n\} \subset \mathcal{M}(\partial\Omega)$ be a bounded sequence such that $|\nu_n| \in \mathcal{M}^g(\partial\Omega)$ for every n . Denote by u_n , $u_{n,1}$ and $u_{n,2}$ the solution of (1.1) with $\mu = 0$ and $\nu = \nu_n$, $\nu = \nu_n^+$ and $\nu = -\nu_n^-$ respectively. Assume that*

$$(3.17) \quad \nu_n^+ \rightharpoonup \nu^+ \quad \text{and} \quad \nu_n^- \rightharpoonup \nu^- \quad \text{in } \mathcal{M}(\partial\Omega).$$

Then $\{u_n\}$ converges in $L^1(\Omega)$ if and only if $\{u_{n,1}\}$ and $\{u_{n,2}\}$ converge in $L^1(\Omega)$. Assuming the convergence of these sequences, denote by $\nu^\#$, $\nu_1^\#$ and $\nu_2^\#$ the reduced limits of $\{\nu_n\}$, $\{\nu_n^+\}$ and $\{-\nu_n^-\}$ respectively. Then

$$(3.18) \quad \nu^\# = (\nu^\#)^+, \quad \nu_2^\# = -(\nu^\#)^-.$$

In particular

$$\nu^\# = \nu_1^\# + \nu_2^\#$$

and

$$\nu^\# = \nu \text{ if and only if } \nu_1^\# = \nu^+ \text{ and } \nu_2^\# = -\nu^-.$$

Proof. First assume that $\{u_n\}$, $\{u_{n,1}\}$ and $\{u_{n,2}\}$ converge in $L^1(\Omega)$. In that case, (3.18) is proved exactly in the same way as [3, Proposition 7.3], using Lemma 3.5 and the last assertion of Theorem 3.4.

Next assume that $\{u_n\}$ converges in $L^1(\Omega)$ and let $\nu^\#$ be the reduced limit of $\{\nu_n\}$. Extract a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}^+\}$ and $\{u_{n_k}^-\}$ converge in $L^1(\Omega)$. Denote the limits of these sequences by u' and u'' respectively. By (3.18)

$$\text{tr } u' = \nu_1^\# = (\nu^\#)^+.$$

Thus u' is independent of the subsequence previously extracted. This implies that $u_n^+ \rightarrow u'$ in $L^1(\Omega)$. Similarly we conclude that $u_n^- \rightarrow u''$ in $L^1(\Omega)$.

The same argument shows that if $\{u_{n,1}\}$ and $\{u_{n,2}\}$ converge in $L^1(\Omega)$ then $\{u_n\}$ converges in $L^1(\Omega)$. \square

As a consequence of this proposition one obtains the following extension of Theorem 3.6 to sequences of signed measures.

Corollary 3.8. *In addition to the assumptions of Proposition 3.7 assume that g satisfies (3.15). Let $\bar{\nu}_n = \mathbb{P}(|\nu_n|)$ and assume that $\{g \circ \bar{\nu}_n\}$ is bounded in $L^1(\Omega; \rho)$. Then $\nu^\#$ and ν are mutually absolutely continuous. More precisely, $(\nu^\#)^+$ and ν^+ (respectively $(\nu^\#)^-$ and ν^-) are mutually a.c.*

4. Reduced limit of a sequence of pairs in $\mathcal{M}^g(\bar{\Omega})$

In this section we discuss the reduced limit of a sequence of pairs $\{(\mu_n, \nu_n)\} \subset \mathcal{M}^g(\bar{\Omega})$ associated with problem,

$$(4.1) \quad \begin{aligned} \Delta u_n + g \circ u_n &= \mu_n & \text{in } \Omega, \\ u_n &= \nu_n & \text{on } \partial\Omega. \end{aligned}$$

We assume that ν_n satisfies (3.2) and μ_n satisfies

$$(4.2) \quad \begin{aligned} (i) \quad & \mu_n \text{ is a good measure in } \Omega \quad \forall n \in \mathbb{N} \\ (ii) \quad & \rho \mu_n \xrightarrow{\bar{\Omega}} \tau \in \mathcal{M}(\bar{\Omega}) \quad (\text{see Definition 3.2}). \end{aligned}$$

Theorem 4.1. *Assume that $g \in \mathcal{G}_0$, $(\mu_n, \nu_n) \in \mathcal{M}^g(\bar{\Omega})$, $\{\nu_n\}$ satisfies (3.2) and $\{\mu_n\}$ satisfies (4.2). Let u_n be the solution of (4.1) and assume that*

$$u_n \rightarrow u \quad \text{in } L^1(\Omega).$$

Then:

- (i) $\{\rho(g \circ u_n)\}$ converges weakly in $\bar{\Omega}$ and
- (ii) $\exists \mu^* \in \mathcal{M}(\Omega, \rho)$, $\nu^* \in \mathcal{M}(\partial\Omega)$ such that

$$(4.3) \quad \begin{aligned} -\Delta u + g \circ u &= \mu^* & \text{in } \Omega \\ u &= \nu^* & \text{on } \partial\Omega. \end{aligned}$$

Furthermore, if $\mu_n \geq 0$ and $\nu_n \geq 0$ for every $n \geq 1$ then

$$(4.4) \quad 0 \leq \nu^* \leq (\nu + \tau \mathbf{1}_{\partial\Omega}).$$

Remark. By [3, Theorem 1.3], μ^* is independent of ν_n .

Proof. Our assumptions imply that $\{\nu_n\}$ is bounded in $\mathcal{M}(\partial\Omega)$ and $\{\mu_n\}$ is bounded in $\mathcal{M}(\Omega; \rho)$. Hence $\{\rho(g \circ u_n)\}$ is bounded in $L^1(\Omega)$. Therefore there exists a subsequence (still denoted by $\{u_n\}$) such that

$$\rho g \circ u_n \xrightarrow{\bar{\Omega}} \lambda$$

(see Definition 3.2). Put

$$\lambda_{int} = \frac{\lambda}{\rho} \mathbf{1}_{\Omega} \text{ and } \lambda_{bd} = \lambda \mathbf{1}_{bd}.$$

By Lemma 3.3,

$$(4.5) \quad \lim_{n \rightarrow \infty} \int_{\Omega} (g \circ u_n) \varphi dx = \int_{\Omega} \varphi d\lambda_{int} - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d\lambda$$

and (3.4) holds for every $\varphi \in C_0^2(\bar{\Omega})$.

As u_n is the weak solution of (4.1),

$$(4.6) \quad \int_{\Omega} (-u_n \Delta \varphi + (g \circ u_n) \varphi) dx = \int_{\Omega} \varphi d\mu_n - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d\nu_n$$

for every $\varphi \in C_0^2(\bar{\Omega})$. Taking the limit as $n \rightarrow \infty$ and using (3.4) and (4.5) we obtain,

$$- \int_{\Omega} u \Delta \varphi dx + \int_{\Omega} \varphi d(\lambda_{int} - \mu_{int}) - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d(\lambda_{bd} - \tau_{bd}) = - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d\nu$$

for every $\varphi \in C_0^2(\bar{\Omega})$. Thus u is the weak solution of (4.3) where

$$(4.7) \quad \mu^* = g \circ u - (\lambda_{int} - \mu_{int})$$

$$(4.8) \quad \nu^* := \nu - (\lambda_{bd} - \tau_{bd}).$$

By [3, Theorem 1.3], μ^* depends on $\{\mu_n\}$ but is independent of $\{\nu_n\}$.

The fact that u is the weak solution of (4.3) implies that ν^* is the M-boundary trace of u ; as such ν^* is independent of the specific weakly convergent subsequence of $\{\rho(g \circ u_n)\}$. Therefore, by (4.8), λ_{bd} is independent of the subsequence. In addition by (4.7) and [3, Theorem 1.3], λ_{int} is independent of the subsequence. This implies that the full sequence $\{\rho(g \circ u_n)\}$ converges to λ .

If $\mu_n, \nu_n \geq 0$ then $u_n \geq 0$ and $g \circ u_n \geq 0$. Therefore, in this case, $\nu^* \geq 0$ and $\lambda \geq 0$; hence, by (4.8) $\nu^* \leq \nu + \tau_{bd}$. \square

Definition 4.2. If $\{(\mu_n, \nu_n)\} \in \mathcal{M}^g(\bar{\Omega})$, $\{\nu_n\}$ satisfies (3.2) and $\{\mu_n\}$ satisfies (4.2) we say that $\{(\mu_n, \nu_n)\}$ is a g -good sequence that converges weakly to (τ, ν) in $\bar{\Omega}$.

If in addition $u_n \rightarrow u$ in $L^1(\Omega)$ we say that (μ^*, ν^*) , defined as in Theorem 4.1, is the reduced limit and ν^* is the boundary reduced limit of $\{(\mu_n, \nu_n)\}$.

Theorem 4.3. *In addition to the assumptions of Theorem 4.1, assume that $g(x, \cdot)$ satisfies (3.15).*

Let v_n be weak solution of

$$(4.9) \quad -\Delta v_n = \mu_n \quad \text{in } \Omega, \quad v_n = \nu_n \quad \text{on } \partial\Omega.$$

If $\mu_n, \nu_n \geq 0$ and $\{g \circ v_n\}$ is bounded in $L^1(\Omega; \rho)$ then ν^ (defined as in Theorem 4.1) and $\nu + \mu_{bd}$ are absolutely continuous with respect to each other.*

Remark. As a consequence of [3, Theorem 8.1] in combination with [3, Theorem 1.3], μ^* and μ_{int} are absolutely continuous with respect to each other.

Proof. Given $\alpha \in (0, 1)$, we have

$$0 \leq g \circ (\alpha v_n) \leq g \circ v_n.$$

Thus there exists $C_0 > 0$ such that

$$\|g \circ (\alpha v_n)\|_{L^1(\Omega, \rho)} \leq C_0 \quad \forall n \geq 1, \quad \forall \alpha \in (0, 1).$$

Let $\{\alpha_k\}$ be a sequence decreasing to zero. One can extract a subsequence of $\{\rho(g \circ (\alpha v_n))\}$ (still denoted $\{\rho(g \circ (\alpha v_n))\}$) such that, for each k , there exists a measure $\sigma_k \in \mathcal{M}(\bar{\Omega})$ such that

$$(4.10) \quad \rho g \circ (\alpha_k v_n) \xrightarrow{\bar{\Omega}} \sigma_k.$$

Let $w_{n,k}$ be the solution of the problem

$$(4.11) \quad \begin{aligned} -\Delta w + g \circ w &= \alpha_k \mu_n \quad \text{in } \Omega, \\ w &= \alpha_k \nu_n \quad \text{on } \partial\Omega. \end{aligned}$$

$\alpha_k v_n$ is a supersolution of problem (4.11); therefore

$$(4.12) \quad w_{n,k} \leq \alpha_k v_n.$$

Passing to a subsequence, we may assume that $\{w_{n,k}\}$ converges in $L^1(\Omega)$ for each $k \in \mathbb{N}$. Denote by (μ_k^*, ν_k^*) the reduced limit of $(\alpha_k \mu_n, \alpha_k \nu_n)$. By Theorem 4.1 $\{\rho(g \circ w_{n,k})\}$ converges weakly in $\bar{\Omega}$ for each $k \in \mathbb{N}$; we denote its limit by λ_k . By the proof of Theorem 4.1—specifically (4.8)—

$$\nu_k^* = \alpha_k \nu - (\lambda_k - \alpha_k \tau) \mathbf{1}_{\partial\Omega}.$$

By (4.12)

$$\rho(g \circ (\alpha_k v_n) - \rho(g \circ w_{n,k})) \xrightarrow{\bar{\Omega}} \sigma_k - \lambda_k \geq 0.$$

Thus

$$(4.13) \quad (\sigma_k - \lambda_k) \mathbf{1}_{\partial\Omega} = \sigma_k \mathbf{1}_{\partial\Omega} + \nu_k^* - \alpha_k (\nu + \tau \mathbf{1}_{\partial\Omega}) \geq 0.$$

Let u_n be the solution of (4.1). Evidently $w_{n,k} \leq u_n$ for every $k, n \in \mathbb{N}$. Consequently

$$w_k := \lim w_{n,k} \leq \lim u_n = u.$$

This in turn implies that

$$(4.14) \quad \nu_k^* = \text{tr } w_k \leq \text{tr } u \leq \nu^*.$$

Finally, combining (4.13) and (4.14) we obtain

$$(4.15) \quad \alpha_k(\nu + \tau \mathbf{1}_{\partial\Omega}) \leq \sigma_k \mathbf{1}_{\partial\Omega} + \nu^*.$$

In view of (3.15), for every $\epsilon > 0$ there exist $a_0, t_0 > 1$, such that

$$(4.16) \quad \frac{g(x, at)}{ag(x, t)} \geq \frac{1}{\epsilon} \quad \forall a \geq a_0, \quad t \geq t_0.$$

Consider the splitting of $\rho(g \circ (\alpha_k v_n))$ as follows,

$$\rho(g \circ (\alpha_k v_n)) = \rho(g \circ (\alpha_k v_n)) \mathbf{1}_{[\alpha_k v_n < t_0]} + \rho(g \circ (\alpha_k v_n)) \mathbf{1}_{[\alpha_k v_n \geq t_0]}.$$

By passing to a subsequence we may assume that each of the terms on the right hand side converges weakly in $\bar{\Omega}$ to $\sigma_{1,k}$ and $\sigma_{2,k}$ respectively, for each $k \geq 1$. Since $\{\rho(g \circ (\alpha_k v_n)) \mathbf{1}_{[\alpha_k v_n < t_0]}\}$ is uniformly bounded, $\sigma_{1,k} \mathbf{1}_{\partial\Omega} = 0$. Thus

$$\sigma_k \mathbf{1}_{\partial\Omega} = \sigma_{2,k} \mathbf{1}_{\partial\Omega}.$$

But

$$\|\sigma_{2,k}\|_{\mathcal{M}(\bar{\Omega})} \leq \liminf_{n \rightarrow \infty} \int_{[\alpha_k v_n \geq t_0]} \rho(g \circ (\alpha_k v_n)).$$

Therefore

$$\|\sigma_k \mathbf{1}_{\partial\Omega}\|_{\mathcal{M}(\partial\Omega)} \leq \liminf_{n \rightarrow \infty} \int_{[\alpha_k v_n \geq t_0]} \rho(g \circ (\alpha_k v_n)).$$

For k sufficiently large, say $k \geq k_\epsilon$, $\frac{1}{\alpha_k} \geq a_0$. Applying (4.16) with $a = \frac{1}{\alpha_k}$, $t = \alpha_k v_n$, we obtain

$$g \circ (\alpha_k v_n) \mathbf{1}_{[\alpha_k v_n \geq t_0]} \leq \alpha_k \epsilon (g \circ v_n)$$

for $k \geq k_\epsilon$ and $n \geq 1$. Hence

$$\|\sigma_k \mathbf{1}_{\partial\Omega}\|_{\mathcal{M}(\partial\Omega)} \leq \alpha_k \epsilon \liminf_{n \rightarrow \infty} \int_{\Omega} \rho(g \circ v_n) \leq C_0 \epsilon \alpha_k \quad \forall k \geq k_\epsilon.$$

Therefore

$$(4.17) \quad \frac{\|\sigma_k \mathbf{1}_{\partial\Omega}\|_{\mathcal{M}(\partial\Omega)}}{\alpha_k} \rightarrow 0.$$

Since $\nu^* \leq \nu + \tau \mathbf{1}_{\partial\Omega}$, we only have to prove that $\nu + \tau \mathbf{1}_{\partial\Omega}$ is absolutely continuous with respect to ν^* . Let $E \subset \partial\Omega$ be a Borel set such that $\nu^*(E) = 0$. Then, by (4.15)

$$\alpha_k(\nu(E) + \tau(E)) \leq \sigma_k(E) \quad \forall k \geq 1.$$

This inequality and (4.17) imply that $\nu(E) + \tau(E) = 0$. \square

Lemma 4.4. *Let $g \in \mathcal{G}_0$. Assume that $\{(\mu_n, \nu_n)\}$ and $\{(\tilde{\mu}_n, \tilde{\nu}_n)\}$ be g good sequences converging weakly in $\bar{\Omega}$ to (τ, ν) and $(\tilde{\tau}, \tilde{\nu})$ respectively.*

Let u_n (resp \tilde{u}_n) be the solution of (1.1) with $(\mu, \nu) = (\mu_n, \nu_n)$ (resp. $(\tilde{\mu}_n, \tilde{\nu}_n)$). Assume that

$$u_n \rightarrow u, \quad \tilde{u}_n \rightarrow \tilde{u} \quad \text{in } L^1(\Omega)$$

and let (μ^*, ν^*) and $(\tilde{\mu}^*, \tilde{\nu}^*)$ denote the reduced limits of $\{(\mu_n, \nu_n)\}$ and $\{(\tilde{\mu}_n, \tilde{\nu}_n)\}$ respectively.

Under these assumptions, if

$$\mu_n \leq \tilde{\mu}_n, \quad \nu_n \leq \tilde{\nu}_n \quad \forall n \geq 1$$

then

$$(4.18) \quad \begin{aligned} (a) \quad & 0 \leq \tilde{\nu}^* - \nu^* \leq (\tilde{\nu} - \nu) + (\tilde{\tau} - \tau)\mathbf{1}_{\partial\Omega}, \\ (b) \quad & 0 \leq \tilde{\mu}^* - \mu^* \leq \frac{1}{\rho}(\tilde{\tau} - \tau)\mathbf{1}_{\Omega} =: \tilde{\mu}_{int} - \mu_{int}. \end{aligned}$$

Proof. Inequality (4.18) (b) is proved in [3, Theorem 7.1]. (Recall that the reduced limit μ^* is independent of $\{\nu_n\}$.) It remains to prove (4.18)(a). Clearly $u_n \leq \tilde{u}_n$, thus $u \leq \tilde{u}$. Hence $\nu^* \leq \tilde{\nu}^*$. By Theorem 4.1 there exist measures $\lambda, \tilde{\lambda} \in \mathcal{M}(\bar{\Omega})$ such that

$$\rho g \circ u_n \rightharpoonup_{\bar{\Omega}} \lambda \quad \text{and} \quad \rho g \circ \tilde{u}_n \rightharpoonup_{\bar{\Omega}} \tilde{\lambda}.$$

Since $u_n \leq \tilde{u}_n$, we also have $\lambda \leq \tilde{\lambda}$. Therefore from Theorem 4.1

$$\begin{aligned} 0 \leq \tilde{\nu}^* - \nu^* &= \tilde{\nu} + \tilde{\tau}\mathbf{1}_{\partial\Omega} - \tilde{\lambda}\mathbf{1}_{\partial\Omega} - (\nu + \tau\mathbf{1}_{\partial\Omega} - \lambda\mathbf{1}_{\partial\Omega}) \\ &= (\tilde{\nu} - \nu) + (\tilde{\tau} - \tau)\mathbf{1}_{\partial\Omega} - (\tilde{\lambda} - \lambda)\mathbf{1}_{\partial\Omega} \\ &\leq (\tilde{\nu} - \nu) + (\tilde{\tau} - \tau)\mathbf{1}_{\partial\Omega} \end{aligned}$$

This proves (4.18)(a). \square

Corollary 4.5. Let $g \in \mathcal{G}_0$, u_n be the weak solution of (4.1) and v_n be the weak solution of (1.1) with $(\mu, \nu) = (\tilde{\mu}_n, \nu_n)$. Assume that

$$(4.19) \quad \begin{aligned} \rho\mu_n &\rightharpoonup_{\bar{\Omega}} \mu, \quad \rho\tilde{\mu}_n \rightharpoonup_{\bar{\Omega}} \tilde{\mu} \quad \text{and} \quad \nu_n \rightharpoonup \nu; \\ u_n &\rightarrow u, \quad v_n \rightarrow v \quad \text{in} \quad L^1(\Omega). \end{aligned}$$

Let (μ^*, ν^*) (respectively $(\tilde{\mu}^*, \tilde{\nu}^*)$) denote the reduced limit of $\{(\mu_n, \nu_n)\}$ (respectively $\{(\tilde{\mu}_n, \nu_n)\}$). If $\mu_n \leq \tilde{\mu}_n$ and $\{\tilde{\mu}_n - \mu_n\}$ is tight then $\nu^* = \tilde{\nu}^*$.

Proof. By Lemma 4.4,

$$0 \leq \tilde{\nu}^* - \nu^* \leq (\tilde{\tau} - \tau)\mathbf{1}_{\partial\Omega}.$$

Since $\{\tilde{\mu}_n - \mu_n\}$ is tight we have $(\tilde{\tau} - \tau)\mathbf{1}_{\partial\Omega} = 0$ and consequently $\nu^* = \tilde{\nu}^*$. \square

The next corollary provides an improved inequality for ν^* (compare to (4.4)).

Corollary 4.6. Let $\{(\mu_n, \nu_n)\}$ be a g -good sequence weakly convergent to (τ, ν) in $\bar{\Omega}$ (in the sense of Definition 4.2). Assume that the sequence has reduced limit (μ^*, ν^*) .

If $\mu_n \geq 0$ and $\nu_n \geq 0$ for every $n \geq 1$ then

$$(4.20) \quad \nu^\# \leq \nu^* \leq \nu^\# + \tau\mathbf{1}_{\partial\Omega},$$

where $\nu^\#$ is the reduced limit of $\{\nu_n\}$ defined in Section 2.

Proof. We apply Lemma 4.4 to the sequences $\{(\mu_n, \nu_n)\}$ and $\{(0, \nu_n)\}$ \square

5. Subcritical problem

Theorem 5.1. *Assume that $g \in \mathcal{G}_0$ has subcritical growth with respect to the boundary, i.e., there exists $C > 0$ and $q < \frac{N+1}{N-1}$ such that*

$$(5.1) \quad |g(x, t)| \leq C(|t|^q + 1) \quad \forall t \in \mathbb{R}.$$

Let $\{\mu_n\} \subset \mathfrak{M}(\Omega; \rho)$ and $\{\nu_n\} \subset \mathfrak{M}(\partial\Omega)$ and let u_n be the weak solution of the problem

$$(5.2) \quad \begin{aligned} -\Delta u_n + g \circ u_n &= \mu_n & \text{in } \Omega, \\ u_n &= \nu_n & \text{on } \partial\Omega. \end{aligned}$$

Assume that

$$(5.3) \quad \nu_n \rightharpoonup \nu \text{ weakly in } \partial\Omega, \quad \rho\mu_n \xrightarrow[\bar{\Omega}]{} \tau \text{ weakly in } \bar{\Omega}.$$

If $u_n \rightarrow u$ in $L^1(\Omega)$ then u is a weak solution of the problem

$$(5.4) \quad \begin{aligned} -\Delta u + g \circ u &= \mu_{int} & \text{in } \Omega, \\ u &= \nu + \tau \mathbf{1}_{\partial\Omega} & \text{on } \partial\Omega. \end{aligned}$$

where $\mu_{int} = \frac{\tau}{\rho} \mathbf{1}_{\Omega}$.

Remark. In the present case, if μ_n, ν_n satisfy the assumptions of the theorem then $\{u_n\}$ has a subsequence converging in $L^1(\Omega)$. This is proved as in Section 2.

Notation: Given $\mu \in \mathfrak{M}(\Omega; \rho)$ we denote by $\mathbb{G}(\mu)$, the weak solution of the problem

$$(5.5) \quad -\Delta u = \mu \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega.$$

Given $\nu \in \mathfrak{M}(\partial\Omega)$ we denote by $\mathbb{P}(\nu)$ the weak solution of the problem

$$(5.6) \quad \Delta v = 0 \quad \text{in } \Omega; \quad v = \nu \quad \text{on } \partial\Omega.$$

Proof. First we show that

$$(5.7) \quad g \circ u_n \rightarrow g \circ u \quad \text{in } L^1(\Omega, \rho).$$

Define $\mathbb{G}(|\mu_n|) := v_n$ and $\mathbb{P}(|\nu_n|) := v'_n$. Then $v_n + v'_n$ satisfies

$$-\Delta(v_n + v'_n) = |\mu_n| \quad \text{in } \Omega; \quad v_n + v'_n = |\nu_n| \quad \text{on } \partial\Omega.$$

Let U_n denote the weak solution of (1.1) with $(\mu, \nu) = (|\mu_n|, |\nu_n|)$. (Condition (5.1) implies that every pair of measures is good.) By comparison principle we have

$$|u_n| \leq U_n \leq v_n + v'_n \quad a.e.$$

Thus

$$|g \circ u_n| \leq g \circ U_n \leq g \circ (v_n + v'_n) \leq C(|v_n + v'_n|^q + 1) \leq C'(|v_n|^q + |v'_n|^q + 1).$$

By classical estimates

$$\|\mathbb{G}(|\mu_n|)\|_{L^p(\Omega; \rho)} \leq c_p \|\mu_n\|_{\mathcal{M}(\Omega; \rho)} \quad \forall p \in [1, (N+1)/(N-1))$$

and

$$\|\mathbb{P}(|\nu_n|)\|_{L^p(\Omega;\rho)} \leq c_p \|\nu_n\|_{\mathcal{M}(\partial\Omega)} \quad \forall p \in [1, (N+1)/(N-1)).$$

Hence, $\{v_n\}$ and $\{v'_n\}$ are bounded in $L^p(\Omega;\rho)$ for every p as above. This in turn implies that they are uniformly integrable in each of these spaces. It follows that $\{g \circ u_n\}$ is uniformly integrable in $L^1(\Omega;\rho)$. Since $u_n \rightarrow u$ in $L^1(\Omega)$ there exists a subsequence $\{u_{n_k}\}$ that converges a.e. to u . Therefore $g \circ u_{n_k} \rightarrow g \circ u$ in $L^1(\Omega;\rho)$. As the limit does not depend on the subsequence we conclude that $g \circ u_n \rightarrow g \circ u$ in $L^1(\Omega;\rho)$.

By (5.7) $\{g \circ u_n\}$ is bounded in $\mathcal{M}(\Omega;\rho)$; therefore a subsequence (still denoted $\{g \circ u_n\}$) converges weakly in $\mathcal{M}(\bar{\Omega})$ to a measure λ . As u_n is a weak solution of (5.2),

$$(5.8) \quad \int_{\Omega} (-u_n \Delta \varphi + (g \circ u_n) \varphi) dx = \int_{\Omega} \varphi d\mu_n - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d\nu_n \quad \forall \varphi \in C_0^2(\bar{\Omega}).$$

By Lemma 3.3 and (5.7),

$$\int_{\Omega} (g \circ u_n) \varphi dx \rightarrow \int_{\Omega} (g \circ u) \varphi dx$$

and

$$\int_{\Omega} \varphi d\mu_n \rightarrow \int_{\Omega} \varphi d\mu_{int} - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d\tau.$$

Therefore taking the limit in (5.8), we obtain

$$\int_{\Omega} (-u \Delta \varphi + (g \circ u) \varphi) dx = \int_{\Omega} \varphi d\mu_{int} - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \mathbf{n}} d(\nu + \tau \mathbf{1}_{\partial\Omega}).$$

□

6. Negligible measures

Theorem 6.1. *Assume that $g \in \mathcal{G}_0$ is convex and satisfies the Δ_2 condition. Let $\{(\mu_n, \nu_n)\}$ and $\{(\tilde{\mu}_n, \tilde{\nu}_n)\}$ be g -good sequences converging weakly in $\bar{\Omega}$ to (τ, ν) and $(\tilde{\tau}, \tilde{\nu})$ respectively. Assume that, for every $n \geq 1$, $(|\mu_n|, |\nu_n|)$ and $(|\tilde{\mu}_n|, |\tilde{\nu}_n|)$ are in $\mathcal{M}^g(\bar{\Omega})$.*

Let u_n (resp \tilde{u}_n) be the solution of (1.1) with $(\mu, \nu) = (\mu_n, \nu_n)$ (resp. $(\tilde{\mu}_n, \tilde{\nu}_n)$). Assume that

$$u_n \rightarrow u, \quad \tilde{u}_n \rightarrow \tilde{u} \quad \text{in } L^1(\Omega)$$

and let (μ^, ν^*) and $(\tilde{\mu}^*, \tilde{\nu}^*)$ denote the reduced limits of $\{(\mu_n, \nu_n)\}$ and $\{(\tilde{\mu}_n, \tilde{\nu}_n)\}$ respectively.*

Assume that a subsequence of $\{\rho|\tilde{\mu}_n - \mu_n|\}$ converges weakly in $\mathcal{M}(\bar{\Omega})$ to a measure Λ such that $\Lambda \mathbf{1}_{\partial\Omega}$ is negligible. Then

$$(6.1) \quad \nu^* = \tilde{\nu}^*.$$

Proof. First we prove the result in the case that

$$(6.2) \quad \mu_n \leq \tilde{\mu}_n.$$

This condition implies that $u_n \leq \tilde{u}_n$ and consequently $\nu^* \leq \tilde{\nu}^*$. By Lemma 4.4,

$$(6.3) \quad 0 \leq \tilde{\nu}^* - \nu^* \leq (\tilde{\tau} - \tau) \mathbf{1}_{\partial\Omega}.$$

Observe that

$$(6.4) \quad |\nu^*| \in \mathcal{M}^g(\partial\Omega), \quad |\tilde{\nu}^*| \in \mathcal{M}^g(\partial\Omega), \quad \tilde{\nu}^* - \nu^* \in \mathcal{M}^g(\partial\Omega).$$

Passing to a subsequence we may assume that: (a) $\{(|\mu_n|, |\nu_n|)\}$ possesses a reduced limit $(\bar{\mu}, \bar{\nu})$ and (b) $\{\rho|\tilde{\mu}_n - \mu_n|\}$ converges weakly in $\mathcal{M}(\bar{\Omega})$ to a measure Λ such that $\Lambda \mathbf{1}_{\partial\Omega}$ is negligible.

Thus $(\bar{\mu}, \bar{\nu}) \in \mathcal{M}^g(\bar{\Omega})$; since both measures are positive it follows that $\bar{\mu} \in \mathcal{M}^g(\Omega)$ and $\bar{\nu} \in \mathcal{M}^g(\partial\Omega)$. Clearly $|\nu^*| \leq \bar{\nu}$; therefore $|\nu^*| \in \mathcal{M}^g(\partial\Omega)$. Similarly $|\tilde{\nu}^*| \in \mathcal{M}^g(\partial\Omega)$. In view of our assumptions on g , these facts imply that $\tilde{\nu}^* - \nu^* \in \mathcal{M}^g(\partial\Omega)$.

Since $(\tilde{\tau} - \tau) \mathbf{1}_{\partial\Omega}$ is negligible while $\tilde{\nu}^* - \nu^*$ is a non-negative measure in $\mathcal{M}^g(\partial\Omega)$, (6.3) implies that $\nu^* = \tilde{\nu}^*$.

Next we drop condition (6.2). Without loss of generality we may assume that the entire sequence $\{\rho|\tilde{\mu}_n - \mu_n|\}$ converges weakly in $\mathcal{M}(\bar{\Omega})$ to Λ .

Put $\gamma_n := \mu_n + |\tilde{\mu}_n - \mu_n|$. Since g is super additive (as a consequence of the convexity assumption and the fact that $g(x, 0) = 0$) and satisfies the Δ_2 condition $|\gamma_n| \in \mathcal{M}^g(\Omega)$. Since $|\nu_n|$ is a good measure it follows that $(|\gamma_n|, |\nu_n|) \in \mathcal{M}^g(\bar{\Omega})$. Passing to a subsequence we may assume that $\{(\gamma_n, \nu_n)\}$ converges weakly in $\bar{\Omega}$ and possesses a reduced limit (γ^*, ν_1^*) .

Note that

$$\mu_n \leq \gamma_n, \quad \tilde{\mu}_n \leq \gamma_n \quad \forall n \geq 1.$$

Furthermore,

$$\rho(\gamma_n - \mu_n) \rightharpoonup \Lambda.$$

Therefore, by the first part of the proof, applied to the sequences $\{(\gamma_n, \nu_n)\}$ and $\{(\mu_n, \nu_n)\}$ we obtain,

$$\nu^* = \nu_1^*.$$

Next observe that

$$|\gamma_n - \tilde{\mu}_n| \leq 2|\tilde{\mu}_n - \mu_n|.$$

Consider a subsequence of $\{\rho|\gamma_n - \tilde{\mu}_n|\}$ that converges weakly in $\mathcal{M}(\bar{\Omega})$ to a measure Λ' . Then $\Lambda' \leq 2\Lambda$ and, as $\Lambda \mathbf{1}_{\partial\Omega}$ is negligible, it follows that $\Lambda' \mathbf{1}_{\partial\Omega}$ is negligible. Applying the first part of the proof to the sequences $\{(\gamma_n, \nu_n)\}$ and $\{(\tilde{\mu}_n, \nu_n)\}$ we obtain,

$$\tilde{\nu}^* = \nu_1^*.$$

Combining these facts we obtain (6.1). \square

Remark. If all the measures are non-negative and $\mu_n \leq \tilde{\mu}_n$ then the conclusion of the theorem is valid for every $g \in \mathcal{G}_0$, i.e., convexity and the Δ_2 condition are not needed. Indeed in this case ν^* and $\tilde{\nu}^*$ are non-negative and $\nu^* \leq \tilde{\nu}^*$. Furthermore, by definition, the reduced limits belong to $\mathcal{M}^g(\Omega)$. As the measures are non-negative this implies that ν^* and $\tilde{\nu}^*$ are in $\mathcal{M}^g(\partial\Omega)$. These facts imply that $\tilde{\nu}^* - \nu^*$ is a non-negative good measure. As $\tilde{\tau} - \tau$ is negligible, (6.3) implies that $\nu^* = \tilde{\nu}^*$.

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